The University of Nottingham

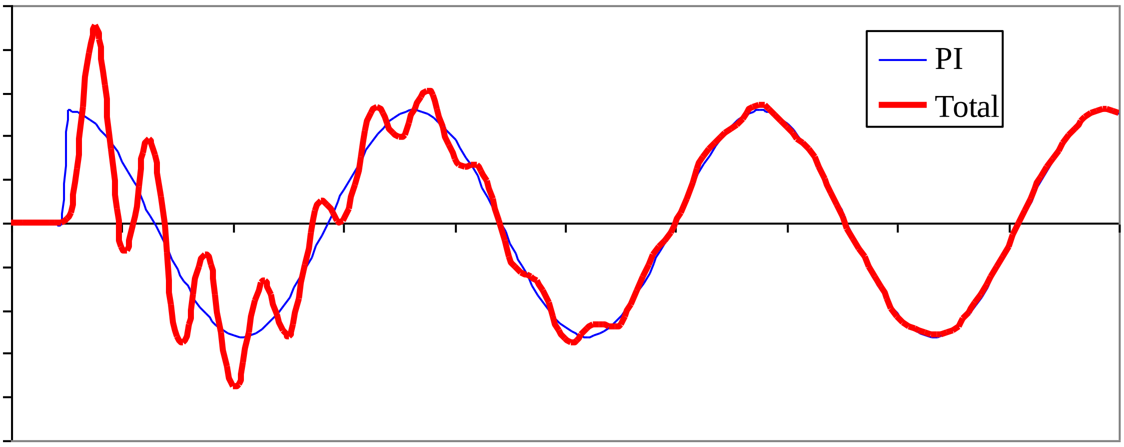
HARMONIC/FORCED EXCITATION OF SINGLE-SEGREE-OF-FREEDOM SYSTEMS

MODULE: MMME2046 DYNAMICS & CONTROL

(i) SOLUTION OF THE EQUATION OF MOTION

$$M\ddot{z} + C\dot{z} + Kz = f(t)$$

The complete solution for $z(t)$ consists of a ***Particular Integral*** and the solution to the ***Complementary Function***. The latter is simply the solution to the equivalent free vibration problem considered in Section A. For non-zero damping, all free vibration solutions tend to zero as time tends to infinity and they describe the ***transient response*** of the structure due to the sudden start of the excitation. In practice, the transient response normally decays quickly. As an example, the graph below shows the complete solution for the response of a single-degree-of-freedom system due to the start of a cosine wave excitation. Near the start, we see the decaying transient response superimposed on the particular integral solution.



After the transient response has decayed, we are left with the particular integral, which continues for as long as the excitation remains. It's normally called the ***stead-state response*** and, in most cases, this is all we are interested in.

METHOD 1 - DIRECT SUBSTITUTION

Consider harmonic excitation of the form, $f(t) = F\cos\omega t$

For pure sinusoidal excitation, the response is also sinusoidal. The response has the same frequency as the excitation, but the two are likely to have a phase difference (see page 2). The key information we want from the analysis is the **amplitude of the response** and its **phase relative to the excitation**. A suitable expression for the response is therefore

$$z(t) = Z\cos(\omega t + \alpha) \qquad (1)$$

where Z is the amplitude of the vibration and α is the phase angle.

To find Z and α, we substitute for $z(t)$ and its derivatives in the equation of motion, expand the various trigonometric terms and equate the coefficients of cos ωt and sin ωt. This gives

$$Z = \frac{F}{\sqrt{\left(K - M\omega^2\right)^2 + \omega^2 C^2}} \qquad (2)$$

and

$$\alpha = \tan^{-1}\left(\frac{-\omega C}{K - M\omega^2}\right) \qquad (3)$$

The following substitution is suggested in the Year 2 Mathematics module.

$$z(t) = A\cos\omega t + B\sin\omega t$$

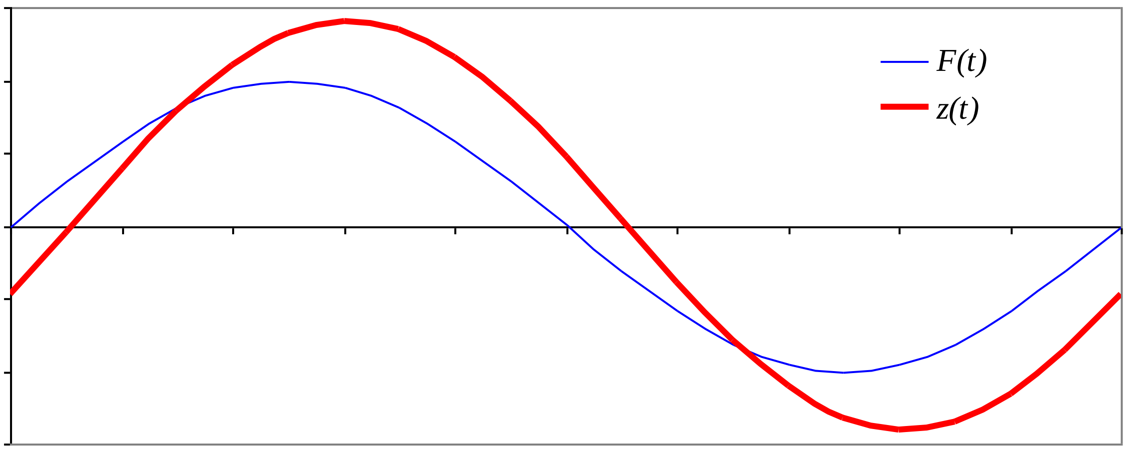
This is an equally valid mathematical substitution. If you choose to use this, you will then need to manipulate the coefficients A and B can to give the required amplitude and phase information.

Q. *What if the excitation had been written in the form* $f(t) = F\sin\omega t$?

To find the amplitude and phase angle information in this case, we would put

$$z(t) = Z\sin(\omega t + \alpha)$$

It is easy to show that the expressions for Z and α are the same as equations (2) and (3).



Since the steady-state response to sinusoidal excitation is also sinusoidal and with the same frequency as the excitation, the key parameters to be identified are the **amplitude** and **phase angle**. Instead of drawing conventional time waveforms, a convenient way of showing the amplitude and phase relationship between excitation and response is with **phasor diagram**.

Method 2 - Complex Algebra

This provides a mathematically convenient way of finding the amplitude and phase angle of the response. It has the added advantage that the same mathematical approach can be extended to more complicated (and therefore more realistic) structures, to more general forms of excitation and to experimental testing and digital data analysis procedures.

The substitutions used are always the same; that is,

Put $$f(t) = F\,e^{i\omega t} \tag{4}$$

and
$$\begin{aligned} z(t) &= Z\,e^{i(\omega t+\alpha)} \\ &= \left(Z\,e^{i\alpha}\right)e^{i\omega t} \\ &= Z^*\,e^{i\omega t} \end{aligned} \tag{5}$$

where Z^* is COMPLEX $\left[= Z\left(\cos\alpha + i\sin\alpha\right)\right]$

Differentiating $z(t)$, $\dot{z} = i\omega\, Z^*\, e^{i\omega t}$ and $\ddot{z} = -\omega^2\, Z^*\, e^{i\omega t}$

When these are substituted into the equation of motion, we get

$$\left(-M\omega^2 + i\omega C + K\right)Z^* e^{i\omega t} = F\,e^{i\omega t}$$

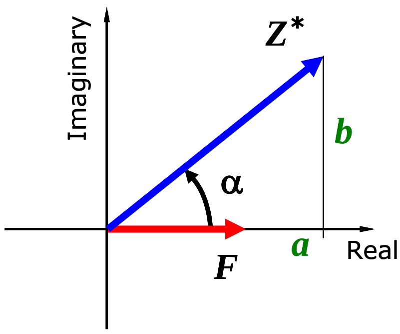
Hence $$Z^* = \frac{F}{\left(K - M\omega^2\right) + i\omega C}$$

or $$Z^* = \frac{F}{\left(K - M\omega^2\right)^2 + \omega^2 C^2}\left[\left(K - M\omega^2\right) - i\omega C\right]$$

or $$Z^* = a + i\,b$$

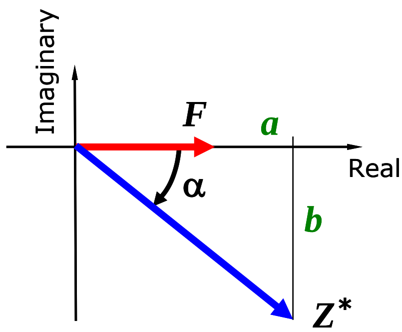
The amplitude, $Z = \left|Z^*\right| = \sqrt{a^2 + b^2}$

and phase angle, $\alpha = \tan^{-1}\left(\dfrac{b}{a}\right)$



Hence, $$Z = \frac{F}{\sqrt{\left(K - M\omega^2\right)^2 + \omega^2 C^2}}$$

and $$\alpha = \tan^{-1}\left(\frac{-\omega C}{K - M\omega^2}\right)$$



Plotting Z^* and F on the complex (Argand) plane, gives the same phasor diagram discussed before. Note that in practice, the imaginary part of Z^* is always negative. As a result, α is also negative, meaning that the response **lags** behind the excitation.

(ii) FREQUENCY CHARACTERISTICS OF THE HARMONIC RESPONSE

To see how the excitation frequency affects the response, we will consider the ***Frequency Response Function*** (FRF). By definition, this is the **Response / Unit Applied Force**. Note that it is used exclusively for **FORCE excitation**.

Start with the general form of the equation of motion,

$$M\ddot{z} + C\dot{z} + Kz = f(t)$$

Dividing by M and noting that $\dfrac{C}{M} = 2\gamma\omega_n$ [*prove this as an exercise*], we get

$$\ddot{z} + 2\gamma\omega_n\dot{z} + \omega_n^2 z = \frac{f(t)}{M}$$

With $f(t) = F\,\mathrm{e}^{\mathrm{i}\omega t}$ and $z(t) = Z^*\mathrm{e}^{\mathrm{i}\omega t}$, $\qquad Z^* = \dfrac{F}{M}\dfrac{1}{\left(\omega_n^2-\omega^2\right)+\mathrm{i}\,2\gamma\omega_n\omega}$

The FRF is therefore: $$H(\omega) = \frac{Z^*}{F} = \frac{1}{M}\frac{1}{\left(\omega_n^2-\omega^2\right)+\mathrm{i}\,2\gamma\omega_n\omega} \tag{6}$$

An alternative expression that emphasises the frequency dependence is obtained by dividing top and bottom by ω_n^2 and noting that $M\omega_n^2 = K$ gives

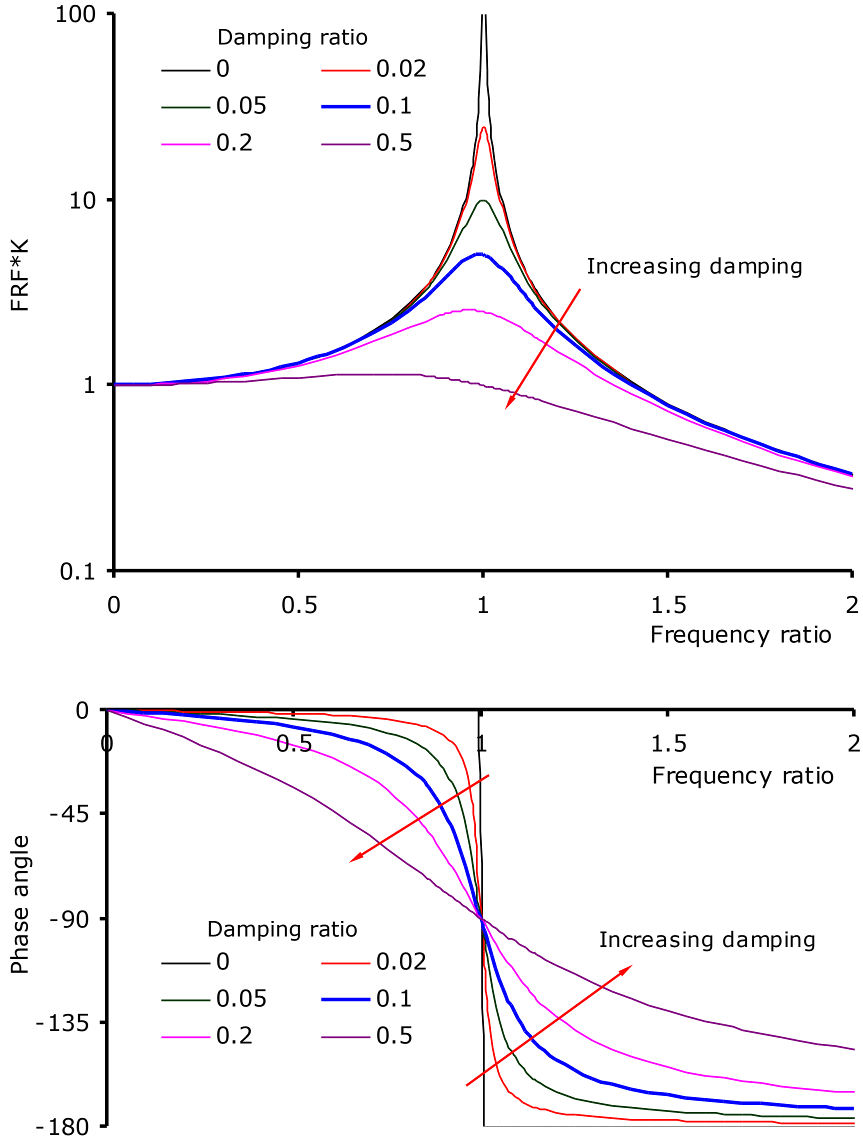
$$H(\omega) = \frac{Z^*}{F} = \frac{1}{K}\frac{1}{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)+\mathrm{i}\,2\gamma\dfrac{\omega}{\omega_n}}$$

$$\text{Magnitude,}\quad |H(\omega)| = \frac{1}{K}\frac{1}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2 + 4\gamma^2\dfrac{\omega^2}{\omega_n^2}}}$$

$$\text{Phase angle},\ \alpha = \tan^{-1}\left(\frac{-2\gamma\dfrac{\omega}{\omega_n}}{1-\dfrac{\omega^2}{\omega_n^2}}\right)$$

It is clear from the above expressions that the response of the structure (its amplitude and phase angle) depends on the ratio between the excitation frequency, ω, and the undamped natural frequency, ω_n and on the damping ratio, γ.

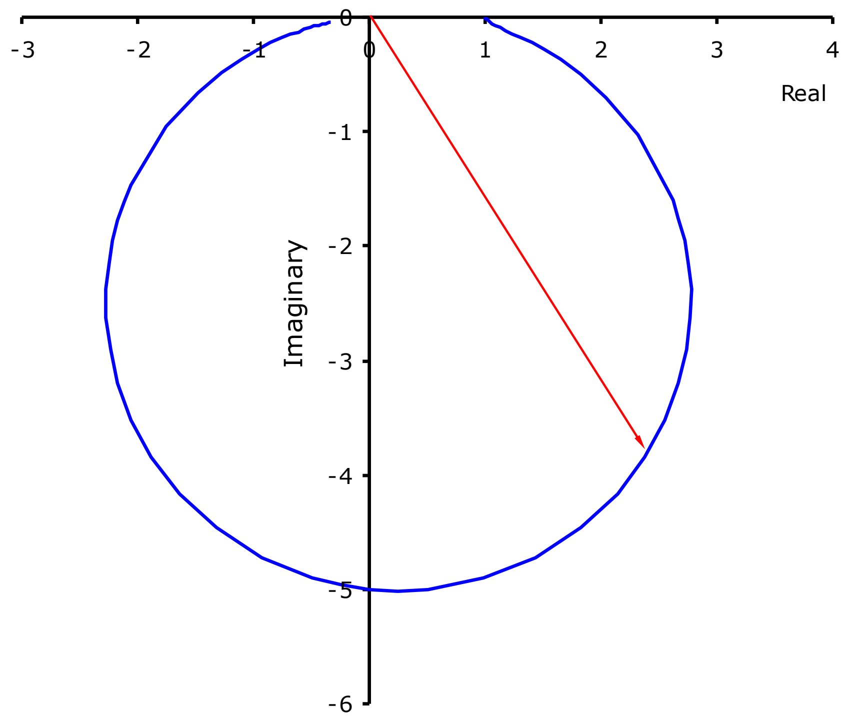
The graphs on the next page show how the FRF amplitude and phase angle vary with both frequency ratio and damping ratio.



Amplitude and phase angle of FRF plotted against frequency ratio

The graphs have been plotted for several values of damping ratio. Most structures have damping ratios of less that 0.1 and it can be seen that the resonant peak (the maximum response near the frequency ratio of 1.0) is large. Increasing damping will reduce the height of this resonant peak[1]. In the phase angle plot, we see that for low values of damping ratio, the phase angle is close to 0° (i.e., the response is roughly in phase with the excitation) for frequencies below the undamped natural frequency and close to -180° (roughly out-of-phase) above the undamped natural frequency.

1 The frequency giving the maximum response is called the ***resonant frequency***. With low damping, the resonant frequency, the undamped natural frequency and the damped natural frequency are all virtually identical. For higher damping (look at the curve for γ = 0.2), the resonant frequency is less than the undamped natural frequency.



Imaginary part of the FRF plotted against the real part (γ = 0.1)

Each point on the real *vs* imaginary plot shown above gives the phasor diagram we met before (the arrow illustrates one particular frequency). Although not shown explicitly, we move around the plot as the frequency ratio varies. Comparing this with the plot on page 5, when $\omega = 0$, the amplitude is 1.0 and the phase angle is 0°. When $\omega = \omega_n$, the phase angle is -90° and the response is purely imaginary. As ω tends to infinity, the amplitude tends to zero and the phase angle tends to -180°.

Experimental Modal Analysis

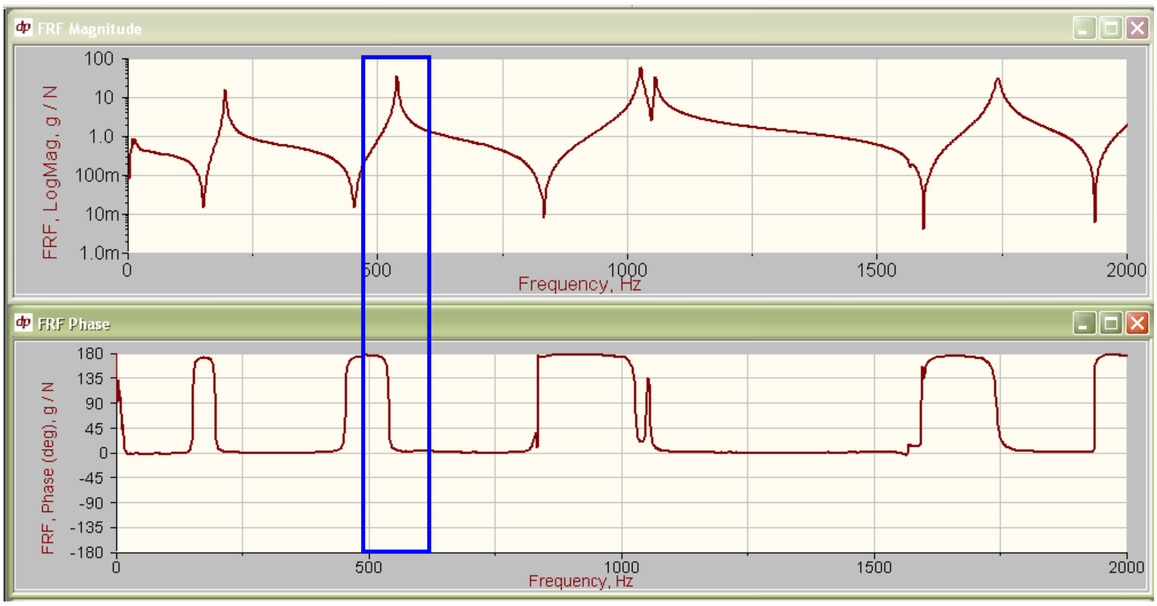
The Frequency Response Function lies at the heart of experimental modal analysis, which aims to measure the natural frequencies, damping and mode shapes of real structures having many modes of vibration.

FRFs can be measured by applying a known force to the structure at any point k, measuring its response at point j and then dividing one by the other. That is:

$$H_{jk}(\omega) = \frac{Z_j^*}{F_k}$$

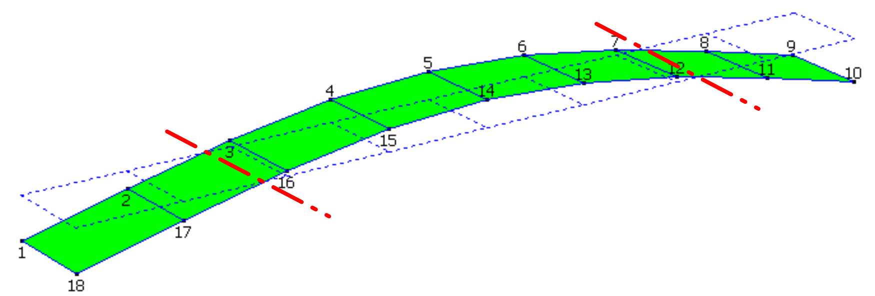
A typical means to implement this in the real world is to utilize "impact testing" in which you will "tap" (relative to size) the structure with a specially instrumented hammer containing a force transducer. An accelerometer attached to a point on the structure will measure the response. This data is processed to give the FRF for that excitation position.

The test structure has many natural frequencies and these will appear as peaks in the FRF amplitude plot. There's an example below showing 5 peaks for a simple beam.



In the vicinity of each natural frequency (one is highlighted in a box above), that particular mode tends to dominate the behaviour. This is why the shapes of the amplitude and phase plots look very similar to the graphs on page 5.

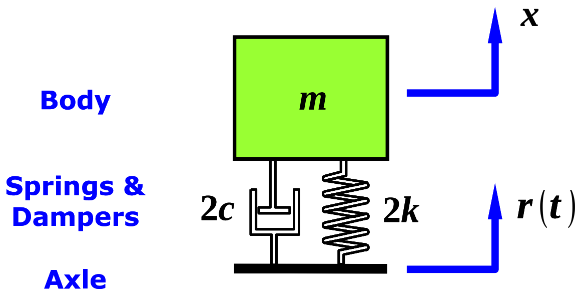
The measured FRF can be further processed to give the natural frequencies, damping and the vibration amplitude and phase for each mode at that test point. In concept, this is done by using an analytical expression for the FRF (similar to equation (6) on page 4) as a fit function to the measured FRF to identify values of ω_n and γ that give the best fit. For a multi-degree-of-freedom system, this fitting process also gives the mode shape amplitude and phase at the test point. By making FRF measurements at several points, the overall deflected shape for each mode can be obtained. As an example, the first mode shape for the test beam is shown below. There are 18 test points here and the variation in vibration amplitude from point to point can be seen. Starting with any of the points, it can also be seen that some of the other points are in phase with it, while others are 180º out of phase.



A more comprehensive introduction to experimental modal analysis and the role of FRFs is given in a document available from the Moodle site.

Worked Examples for Harmonic Excitation

1. Single-Axle Caravan



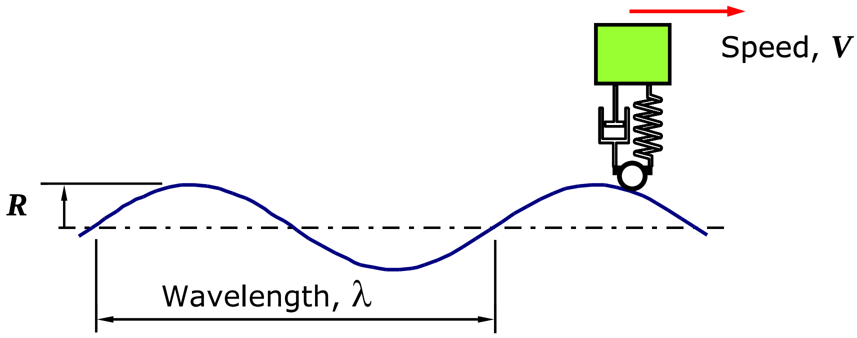
Equation of motion (derived previously) is:

$$m\ddot{x} + 2c\dot{x} + 2kx = 2c\dot{r}(t) + 2k\,r(t) \qquad (1)$$

Suppose the road profile is sinusoidal, so that the displacement input to the axle is:

$$r(t) = R\sin\omega t$$

Q1. How does suspension stiffness affect the response of the caravan?
Q2. Does vehicle speed affect the response?
Q3. How important are the dampers?



The effective excitation frequency depends on the speed of the vehicle and on the wavelength of the road profile. It is given by: $\omega = \dfrac{2\pi V}{\lambda}$ rad/s.

Substitutions for the equation of motion are: $r(t) = R\,e^{\mathbf{i}\omega t}$ and $x(t) = X^*\,e^{\mathbf{i}\omega t}$

Hence from (1),

and $$X^* = \frac{(2k + \mathbf{i}\,2c\omega)R}{(2k - m\omega^2) + \mathbf{i}\,2c\omega}$$

This is in the form: $\dfrac{a + \mathbf{i}b}{c + \mathbf{i}d}$

The amplitude of the response is given by:

$$\left|X^*\right| = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} = \frac{\sqrt{4k^2 + 4c^2\omega^2}\;R}{\sqrt{(2k - m\omega^2)^2 + 4c^2\omega^2}}$$

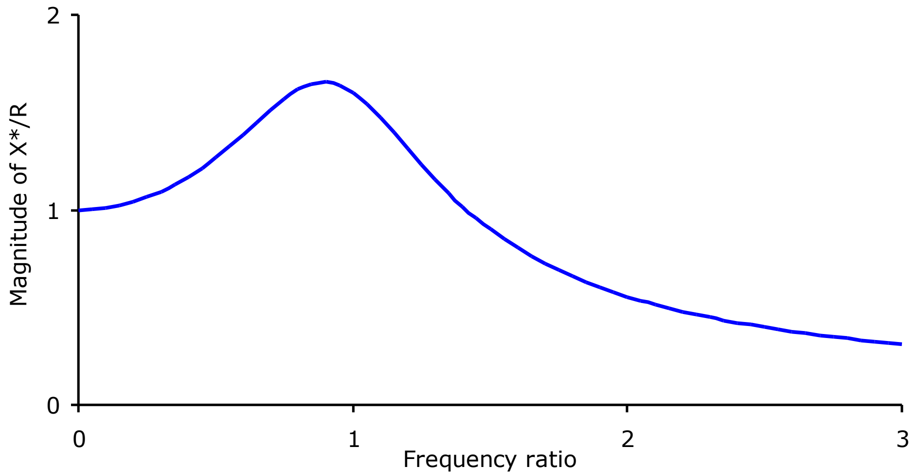
(2)

and the phase angle is: $\alpha = \tan^{-1}\left(\dfrac{-c\,m\,\omega^3}{k\left(2k - m\omega^2\right) + 2c^2\omega^2}\right)$

The graph below shows the variation in response $\left|X^*\right|/R$ with frequency ratio[2], which shows that this is a key factor affecting the response. There are two features to note.

(1) There is a resonant peak at a frequency ratio of about 1. At low frequencies, the amplitude of the caravan is greater than the road input ($\left|X^*\right| > R$). In effect, the suspension is amplifying the road motion.

(2) At higher frequencies (frequency ratios > 1.5), the amplitude of the caravan is less than the road input ($\left|X^*\right| < R$). In this case, the suspension is attenuating the road motion., leading to a smoother ride at higher speeds; the excitation frequency is proportional to the vehicle speed $\left(\omega = \dfrac{2\pi V}{\lambda}\right)$.



So,
Q1. How does suspension stiffness affect the response of the caravan?
Q2. Does vehicle speed affect the response?

(A) What happens if the springs are very stiff? How does speed affect the response?

$$\left|X^*\right| = \frac{\sqrt{4k^2 + 4c^2\omega^2}\;\;R}{\sqrt{\left(2k - m\omega^2\right)^2 + 4c^2\omega^2}} =$$

---

2 As an exercise, show that $\dfrac{\left|X^*\right|}{R} = \dfrac{\sqrt{1 + 4\gamma^2\dfrac{\omega^2}{\omega_n^2}}}{\sqrt{\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)^2 + 4\gamma^2\dfrac{\omega^2}{\omega_n^2}}}$

(B) What happens if the springs are very soft? How does speed affect the response?

What stiffness should the designer choose?

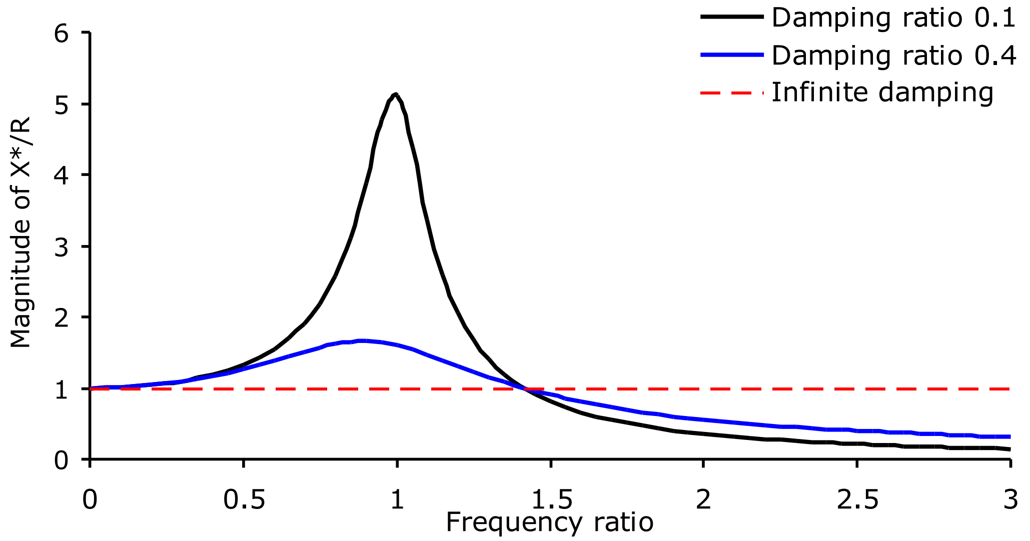
High stiffness generally improves road holding and low stiffness generally improves ride comfort.

Q3. How important are the dampers?

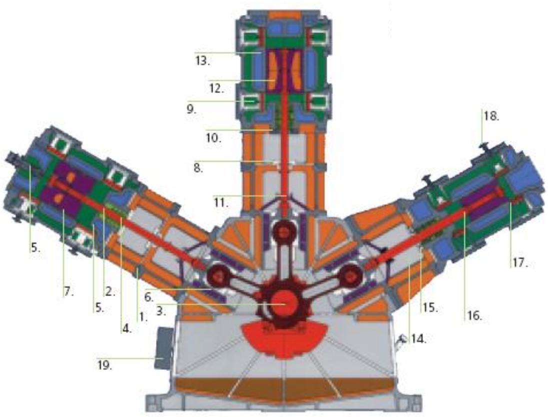
What happens if you remove the dampers?
Does vehicle speed affect the response?

If c = 0, $\left|X^*\right| = \dfrac{\sqrt{4k^2 + 4c^2\omega^2}\;R}{\sqrt{\left(2k - m\omega^2\right)^2 + 4c^2\omega^2}} =$

What damping value should the designer choose?



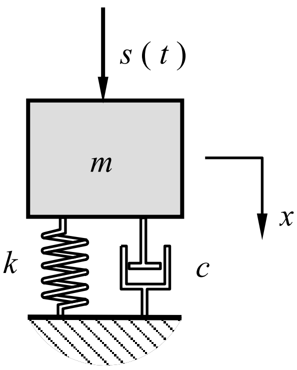
2. Application of the Frequency Response Function



Sketch the waveform of the steady-state vertical displacement of a reciprocating air compressor (mass, m = 4000 kg) operates at a crank speed, Ω, of 300 rev/min. The machine is supported on a set of four resilient mounts, which give an overall vertical stiffness of 2.5 MN/m and a damping ratio of 0.04. The effect of the masses of the reciprocating pistons is to produce a vertical force on the compressor given by

$$s(t) = 1.3\,\Omega^2 \cos\Omega t + 3.0\,\Omega^2 \cos 2\Omega t$$
$$= S_1 \cos\omega_1 t + S_2 \cos\omega_2 t$$

Solution method: The compressor can be modelled as a rigid mass and its mounts as a spring-damper combination. Each sinusoidal term in the excitation will produce a steady-state response that is sinusoidal with the same frequency as the excitation. For example, the term $S_1 \cos\omega_1 t$ will produce a response in the form $X_1 \cos(\omega_1 t + \alpha_1)$. We can use the frequency response function to do this.



Once the response to each excitation term has been found, the total response is obtained by adding the two together.

For each excitation term, we have $X_j^* = H(\omega_j) \times S_j$ so that $X_j = \left|X_j^*\right| = \left|H(\omega_j)\right| \times S_j$ where j = 1 or 2. We can use the expressions on page 4 to work out the FRF and phase angle values.

Using the data given,

S_1 = 1283 N, | $H(\omega_1)$ | = 6.81 x 10^{-4} mm/N, α_1 = -170°, X_1 = 0.87 mm

S_2 = 2961 N, | $H(\omega_2)$ | = 7.52 x 10^{-5} mm/N, α_2 = -178°, X_2 = 0.22 mm

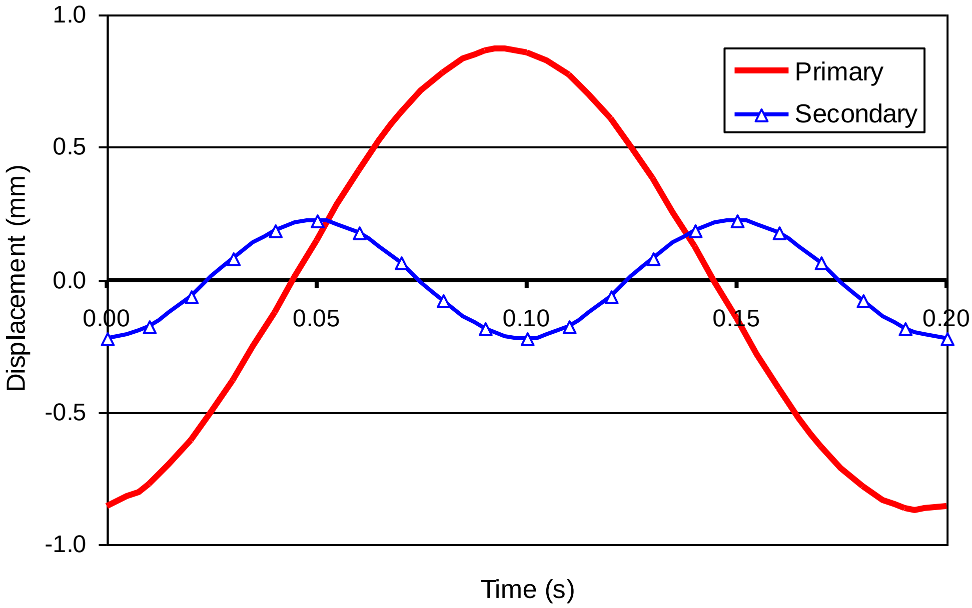
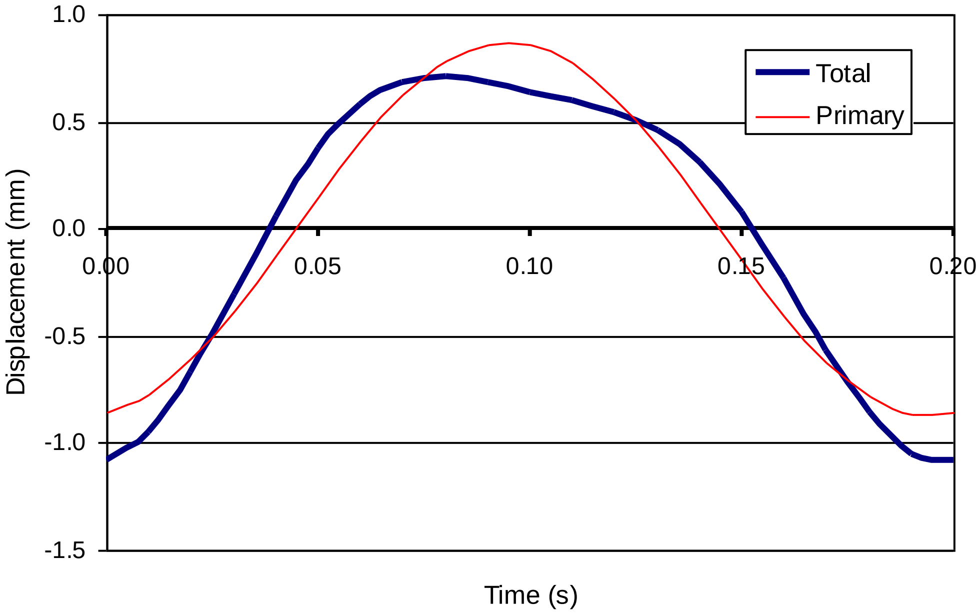


Figure showing $X_1 \cos(\omega_1 t + \alpha_1)$ and $X_2 \cos(\omega_2 t + \alpha_2)$



The thick line shows the result of adding the two waveforms together.

The term "primary" relates to the fact that the vibration frequency, ω_1, is equal to the rotational frequency of the crank. The "secondary" component has a frequency that is twice the rotational frequency of the crank.